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curved, can recede one from the other ever more within a certain finite determinate limit; whence at any rate may arise a suspicion lest the same may be able to happen for two straight lines, unless otherwise demonstrated.

But not therefore,after I in the corollary to the preceding proposition I have made manifest the absolute truth of the aforesaid assumption,is it possible immediately to go over to the assertion of the Euclidean postulate. For previously must also be demonstrated, that those two straights *AH*, *BC*, which with the transversal *AB* make two angles toward the same parts equal to two right angles, as for example each a right angle, do not also, protracted toward these parts to infinity, always separate more from one another beyond all finite assignable distance. For if one chooses to presume the affirmative, which is indeed entirely true in the hypothesis of acute angle; it certainly will not be a legitimate consequence, that the straight *AD* in any way cutting the angle *HAB*, hence of course making at the same time two internal angles *DAB*, *CBA* toward the same parts less than two right angles; that, I say, this straight *AD*, produced to infinity must at length meet with *BC* produced; even if it were at another time demonstrated, that the distance of the two *AH*, *AD* produced to infinity ever greater goes out beyond all finite limit that may be assigned.

But that the aforesaid Clavius should have judged the truth of this assumption sufficient for demonstrating the postulate here in question; that ought to becondoned because of the opinion preconceived by Clavius about equidistant straight lines, which we may discuss more conveniently in a subsequent Scholion.

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INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

(Continued from the July-August Number.)

Construction of Intransitive Groups.

Suppose that we have an intransitive group (G) involving the six letters a, b, c, d, e, f and that in this group a is replaced by b, c, and d but not by e or f. This group must have at least one substitution (s_1) in which a is replaced by b, one (s_2) in which a is replaced by c and one (s_3) in which a is replaced by d. In some power of s_1 (which, from the definition of a group, must also be in G) b is replaced by a.* Let this substitution be denoted by s'_1 and consider the following substitution of G:

$$s'_1\,,\quad s'_1\,s_2\,,\quad s'_1\,s_3\,.$$

In the first of these b is replaced by a, in the second by c and in the third by d. Hence we see that the hypothesis that a is replaced by each of the

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*Suppose s_1 were one of the following substitutions: *ab*, *abc*, *abcd*, *ab.cd*; then b would be replaced by a in the first power of the first and last substitutions, in the second power of the second substitution and the third power of the third.

other letters in the first set of four requires that the other letter of this set or system have the same property, for what we proved in regard to b can be proved, similarly, in regard to c and d.

It is further evident that there can be no substitution in G which replaces one of these four letters by e or f; for suppose that the substitution represented by s_α has this property, replacing, for instance, b by e, then would s_1t replace a by e, which is contrary to hypothesis. Hence e and f can only replace each other in the substitutions of G. Since this proof could clearly be used with respect to any number of letters we have arrived at the following important theorem:

Theorem 1. In every intransitive group the letters may be divided into such systems that the substitutions of the group will connect the letters of the systems transitively and no substitution will replace a letter of one system by one of another system.[†]

Since each of these systems must form a transitive group it follows that *every intransitive group can be formed by combining transitive groups, such that the sum of the letters in these groups is equal to the number of letters in the intransitive group.*

The main problem before us is, therefore, the development of such methods as are most helpful in combining the transitive groups. The two intransitive groups in the list of the groups of four letters are instances of the simplest methods of combination. One of these

$$1,\ ac,\ bd,\ ac.bd$$

is obtained by multiplying every substitution of one group (ac)[*] by every substitution of the other (bd). It is evident that by this method we can always construct an intransitive group from two or more groups each involving different letters, *e. g.* the intransitive group of five letters which may be obtained in this way from (abc) and (de) is

$$1,\ abc,\ acb,\ de,\ abc.de,\ acb.de$$

The other intransitive group in the given list

$$1,\ ab.cd$$

is obtained by the process called simple *isomorphism* or 1, 1 correspondence. The process consists in associating substitutions of the component transitive groups which have the same properties with respect to the groups. The given intransitive group is obtained from the groups

$$\begin{array}{cc} 1 & 1 \\ ab & cd \end{array}$$

It is at once evident that 1 may be associated with 1 and ab with cd, and we thus obtain the required group

$$\begin{array}{c} 1 \\ ab.cd \end{array}$$

[†] cf. Jordan: Traite des Substitutions, Art. 40: also Netto's Theory of Substitutions (Cole's edition), page 70.

[*] The parenthesis is used to indicate the group generated by the substitution enclosed: thus, $(ac)=1,\ ac$; $(abc)=1,\ abc,\ acb$; etc. In case the group consists of more than two substitutions the abbreviation for cyclical is commonly written after the parenthesis: thus, $(abc)=(abc)\ cyc$.

Similarly we may obtain an intransitive group of six letters from (*abc*) and (*def*), viz.

$$1,\ abc.def,\ acb.dfe$$

Instead of a 1, 1 correspondence we may have an *a*, *b* correspondence, *a* and *b* representing any positive integers. In this way we obtain the intransitive group

$$1,\ abc,\ acb,\ ab.de,\ ac.de,\ bc.de$$

from the two groups

$$1,\ abc,\ acb,\ ab,\ ac,\ bc$$

and 1, *de*

by a 3, 1 correspondence.

We have now given the main methods employed in constructing intransitive groups. We proceed to find all the

Intransitive Groups of Five Letters.

All these groups are composed of a transitive group of three and another of two letters: for the only way of dividing five so as to get two or more letters in each system is to divide it into the parts three and two. The component groups are therefore,

$$\begin{array}{l|l} 1,\ abc,\ acb & 1,\ de \\ 1,\ abc,\ acb,\ ab,\ ac,\ bc & \end{array}$$

It is evident that (*de*) can be combined with (*abc*) only by multiplying the two groups together. We thus obtain the intransitive group given above as an illustrative example.

By combining (*de*) with the second group on the left we obtain, in a similar way,

$$\begin{array}{lllll} 1 & abc & abc.de & ab.de & de \\ & acb & acb.de & ac.de & ab \\ & & & bc.de & ac \\ & & & & bc \end{array}$$

In this case the combination may be effected in one more way. Since the first half of the substitutions in the second group on the left form a subgroup we may let 1 of (*de*) correspond to these and *de* to the remaining substitutions. We thus obtain the intransitive group of five letters which was given above as an illustrative example of an *a*, *b* correspondence.

From this we see that there are only three intransitive groups in five letters. In Professor Cayley's list these groups are denoted by

$$(abc)cyc.(de),\ (abc)all(de),\ \{(abc)all(de)\}\text{pos.}$$

There are twenty-one intransitive groups involving six letters. The component groups may involve any of the following systems of letters:

2, 2, 2,
4, 2
3, 3

We expect to apply the given methods to the construction of all these groups. This, it is believed, will give sufficient exercise in the construction of this class of groups and we shall then proceed to the construction of the *transi-*

tive groups.

The reader who desires a thorough working knowledge of this subject could very profitably work over this field and compare his methods and results with those that we shall give.

THE GOLDEN SECTION.

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In the number for November 1892, of *Lehrproben und Lehrgange aus der Praxis der Gymnasien und Realschulen*, there appeared an article by Prof. Dr. O. Willman, entitled Der goldene Schnitt abs ein Thema des mathematischen Unterrichts. The article is interesting not alone to students of geometry, but to all who are at all concerned with the question of concentration, a question which is at present creating for itself an active interest among all educators. The article is a resume of a work on the golden section by F. C. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur, und Kunst, Augsburg, Huttler, 1885. The statements here presented are intended as a summary of the article.

It is very necessary that the connection between instruction in mathematics and in the remaining courses of study should be carefully considered because the subject of mathematics is an abstract one and according to its nature tends towards isolation.

To assist in bringing about this connection, there should be prepared mathematical problems and exercises which will show the application of mathematics to technics and to observations in nature on the one hand, and on the other furnish an insight into the history of mathematics, by means of which historical and classical instruction can be connected with the mathematical. A subject which meets these demands and is at the same time well adapted for purely mathematical instruction is the theme of the golden section, a theme which does not appear in a complete form in our modern text-books.

The simplest division of any magnitude, involving the fewest conditions is the division into two equal parts. Calling a line so divided, S, the parts p, we have $S=2p$, $p=\frac{S}{2}$, $\frac{S}{p}=2$, $\frac{p}{S}=\frac{1}{2}$, $\frac{p}{p}=1$. Contrasted to one case of division into two equal parts stands an infinite number of divisions into two unequal parts; and the ratio of the smaller (m) to the larger (M), $\frac{M}{m}$, or the ratio of one of the parts to the whole, $\frac{m}{S}$ or $\frac{m}{m+M}$ and $\frac{M}{S}$ or $\frac{M}{m+M}$ can be expressed by many different numbers. In one case only is there no need of figures to determine the ratio of the parts to the whole; and that is when